Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-12-01

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to warm-up questions

Question 1. We need to split up the region of integration since we are dealing with a piece-wise defined integrand. The integrand is e^{x^2} when $x \geq y$ and it is e^{y^2} when $x \leq y$. Hence our integral is equal to

$$\int_0^1 \int_0^x e^{x^2}\,\mathrm{d}y\,\mathrm{d}x + \int_0^1 \int_0^y e^{y^2}\,\mathrm{d}x\,\mathrm{d}y$$

which is straightforward to compute.

Question 2.

(a) After drawing a picture, we guess that the flow curves of this vector field are counterclockwise circles centered at the origin. The one passing through $(1,1)$ is then $\mathbf{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t\rangle$, where $\mathbf{r}(\pi/4) = (1,1)$. This is just a guess though; let's actually check that it works:

$$\mathbf{r}'(t) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t\rangle = \mathbf{F}(\sqrt{2}\cos t, \sqrt{2}\sin t)$$

since $\mathbf{F}(x,y) = \langle -y, x\rangle$.

(b) It is possible in general (see the preceding part for instance). However it cannot happen if $\mathbf{F}$ is conservative. This is because if $\mathbf{r}(t)$ is the flow curve C, then $\mathbf{F}\cdot \mathrm{d}\mathbf{r} = |\mathbf{r}'(t)|^2\,\mathrm{d}t$ is always positive along the curve C, but the integral of a conservative vector field along a closed loop is always zero.

Answers to computations

Problem 1.

(a) The Divergence Theorem implies that both integrals are equal to zero, since $\nabla\cdot\mathbf{F} = 0$.

(b) This problem is quite hard. Let's deal with $\iint_{\partial E} |\mathbf{F}\cdot\mathbf{n}|\,\mathrm{d}S$ first, because the cube is easier. It consists of six faces. Consider the top face S_{top}, which has unit normal $\langle 0,0,1\rangle$. The integral over this face is

$$\iint_{S_{\text{top}}} |\langle a,b,c\rangle\cdot\langle 0,0,1\rangle|\,\mathrm{d}S = \iint_{S_{\text{top}}} |c|\,\mathrm{d}S = 16|c|.$$

The other five faces can be handled similarly, and altogether we get

$$32(|a|+|b|+|c|).$$

For the sphere $\iint_{\partial D} |\mathbf{F}\cdot\mathbf{n}|\,\mathrm{d}S$, we just have a single surface, but we should split up the integral based on whether $\mathbf{F}\cdot\mathbf{n}$ is positive or negative. Note that $\mathbf{n}$ is just $\langle x,y,z\rangle$ (e.g. by computing the gradient $\nabla(x^2+y^2+z^2)$ and then rescaling to have length 1). Hence

$$\mathbf{F}\cdot\mathbf{n} = \langle a,b,c\rangle\cdot\langle x,y,z\rangle = ax+by+cz.$$

The plane $ax+by+cz=0$ separates the sphere into two hemispheres. Let S_+ denote the hemisphere where $ax+by+cz\geq 0$, and let S_- denote the hemisphere where $ax+by+cz\leq 0$, both oriented outwards (since we took $\mathbf{n} = \langle x,y,z\rangle$ earlier). We have

$$\iint_{\partial D} |\mathbf{F}\cdot\mathbf{n}|\,\mathrm{d}S = \iint_{S_+} \mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_{S_-} -\mathbf{F}\cdot\mathrm{d}\mathbf{S}.$$

Let D_+ denote the 3D region $x^2+y^2+z^2\leq 1$, $ax+by+cz\geq 0$, i.e. the *solid* hemisphere corresponding to S_+. We have that $\partial(D_+)$ consists of S_+ together with a disk in the plane $ax+by+cz=0$. If we apply the Divergence Theorem to D_+, we obtain

$$\iiint_{D_+} 0\,\mathrm{d}V = \iint_{\text{disk}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_{S_+} \mathbf{F}\cdot\mathrm{d}\mathbf{S}.$$

The disk has radius 1, being an equatorial cross-section of the unit sphere, thus it has area π. So

$$\iint_{\text{disk}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_{\text{disk}} \langle a,b,c\rangle\cdot\frac{-\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}\,\mathrm{d}S = -\pi\sqrt{a^2+b^2+c^2}$$

hence

$$\iint_{S_+} \mathbf{F} \cdot d\mathbf{S} = \pi\sqrt{a^2 + b^2 + c^2}.$$

The integral $\iint_{S_-} -\mathbf{F} \cdot d\mathbf{S}$ can be dealt with similarly, or you can argue by symmetry it must be the same. So the final answer is $2\pi\sqrt{a^2 + b^2 + c^2}$.

One can also approach this problem by taking advantage of the rotational symmetry of the sphere to rotate the vector field into the form $\langle 0, 0, \sqrt{a^2 + b^2 + c^2} \rangle$ for instance. The overall approach would still be the same, but the separating plane just becomes $z = 0$ so things are simpler to write down.

Problem 2.

(a) This is the chain rule.

$$\begin{aligned} \frac{d}{dt}(f(\mathbf{r}(t))) &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\ &= \nabla f \cdot \mathbf{r}'(t) \\ &= D_{\mathbf{r}'(t)} f \end{aligned}$$

where we have used $|\mathbf{r}'(t)| = 1$ in the last step.

(b) Strictly speaking this part does not really rely on the previous, since dz/dt is just the z component of $\mathbf{r}'$. You could interpret it as an application of the previous part in the particular case $f(x, y, z) = z$, in which case $D_{\mathbf{r}'(0)} f$ is just the $\langle 0, 0, 1 \rangle \cdot \mathbf{r}'(0)$, so again, just the z-component of $\mathbf{r}'$.

To find $\mathbf{r}'(0)$, we note that it's tangent to C and that we are told it has length 1. It turns out that, although it is certainly doable, C is very hard to parametrize. It is more convenient to find a tangent vector by noting that such a vector would have to be tangent to both surfaces in the problem, and thus perpendicular to their normal vectors. Hence we take the cross product

$$\langle 1, 1, -2 \rangle \times \langle 3, 4, -5 \rangle = \langle 3, -1, 1 \rangle.$$

Note that $\langle 3, 4, -5 \rangle$ came from evaluating $\nabla(x^2 + y^2 - z^2)$ at the point $(3, 4, 5)$ and then rescaling.

Hence a unit vector in this direction is $\frac{1}{\sqrt{11}}\langle 3, -1, 1 \rangle$. However, this is actually clockwise when viewed from above, so we need to take its negative, meaning that $\mathbf{r}'(0) = -\frac{1}{\sqrt{11}}\langle 3, -1, 1 \rangle$. So the final answer is $-1/\sqrt{11}$.